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Technical Comments

Comment on "Integrated Development of the Equations of Motion for Elastic Hypersonic Flight Vehicles"

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Introduction

BILIMORIA and Schmidt[1] give an integrated development of equations of motion for simulating the trajectory of a vehicle incorporating structural elastic modes, effect of rotating parts, and mass flow rates of air-breathing engines in the equations of motion. This Note shows that the terms containing mass flow rates in the equations of motion of Ref. 1 are erroneous. Further, it is shown that none of the mass-flow-rate terms should be present in the equations of motion because the force and moment attributable to engine thrust are considered as external force and moment acting on the vehicle.

Derivation of Expressions

We first obtain the expressions for force and moment terms attributable to mass flow rate, which exactly match the expressions of Bilimoria and Schmidt. Though some terms in derivation are technically incorrect, they are initially retained just to get comparable expressions. Subsequently, correct expressions are given for force and moment terms along with their physical significance. In the end, reasons are given for totally removing mass-flow-rate terms from the equations of motion.

Let $\bar{R}$ denote the position vector of the origin of body coordinate frame with reference to inertial frame and $\bar{\rho}$ the position of an element of mass dm with reference to origin of body coordinate frame. Then, linear momentum ($\bar{l}$) of the mass element is given by

$$\bar{l} = dm(\dot{\bar{R}} + \dot{\bar{\rho}}) + \underline{d\dot{m}(\bar{R} + \bar{\rho})} \tag{1}$$

$$\text{Force} = \frac{\mathrm{d}\bar{l}}{\mathrm{d}t} = dm(\ddot{\bar{R}} + \ddot{\bar{\rho}}) + 2d\dot{m}(\dot{\bar{R}} + \dot{\bar{\rho}}) + d\ddot{m}(\bar{R} + \bar{\rho}) \tag{2}$$

The underlined term of Eq. (1) is retained only to match the expression of Bilimoria and Schmidt, and it does not have physical significance of linear momentum. In the present context, it is only considered as a part of differentiation of $dm(\bar{R} + \bar{\rho})$ with time. Angular momentum $\bar{h}_I$ and torque $\bar{t}_I$ about the origin of inertial frame are given by

$$\bar{h}_I = (\bar{R} + \bar{\rho}) \times [(\dot{\bar{R}} + \dot{\bar{\rho}})dm + (\bar{R} + \bar{\rho})d\dot{m}]$$

$$= (\bar{R} + \bar{\rho}) \times (\dot{\bar{R}} + \dot{\bar{\rho}})dm$$

$$\text{Torque } \bar{t}_I = \frac{\mathrm{d}\bar{h}_I}{\mathrm{d}t} = (\bar{R} + \bar{\rho}) \times [(\ddot{\bar{R}} + \ddot{\bar{\rho}})dm + (\dot{\bar{R}} + \dot{\bar{\rho}})d\dot{m}]$$

Integration over the entire missile gives

$$\bar{F} = m(\ddot{\bar{R}} + \ddot{\bar{\rho}}) + 2\dot{m}(\dot{\bar{R}} + \dot{\bar{\rho}}) + \ddot{m}(\bar{R} + \bar{\rho}) \tag{3}$$

$$\bar{T}_I = (\bar{R} + \bar{\rho}) \times [(\ddot{\bar{R}} + \ddot{\bar{\rho}})m + \dot{m}(\dot{\bar{R}} + \dot{\bar{\rho}})] \tag{4}$$

Torque about the origin of the body-axes frame is given by

$$\bar{T}_b = \bar{\rho} \times [(\ddot{\bar{R}} + \ddot{\bar{\rho}})m + \dot{m}(\dot{\bar{R}} + \dot{\bar{\rho}})] \tag{5}$$

$\dot{\bar{R}} = (u \;\; v \;\; w)^T$ gives the velocity of the origin of the body frame.

Assuming $\bar{\omega} = (p \;\; q \;\; r)^T$ as the angular velocity of the body frame and

$$\dot{\bar{\rho}} = \frac{\partial \bar{\rho}}{\partial t} + \bar{\omega} \times \bar{\rho} \tag{6}$$

the x component of force and torque attributable to $\dot{m}$ and $\ddot{m}$ terms are obtained as

$$F_x = 2\dot{m}(\dot{x} + qz - ry) + \ddot{m}x + \{2\dot{m}u + \ddot{m}X_I\} \tag{7}$$

$$T_{bx} = \dot{m}[p(y^2 + z^2) + y\dot{z} - \dot{y}z - qxy - rxz] + \{\dot{m}(yw - zv)\} \tag{8}$$

The terms enclosed in braces in Eqs. (7) and (8) are absent in Ref. 1. Other terms when applied at air inlet and fluid outlet match exactly with the expressions given in Tables 1 and 2 of Ref. 1. As stated earlier, the second term in Eq. (1) is not correct because it does not have the physical significance of linear momentum. Thus, the terms containing $\dot{m}$ and $\ddot{m}$ in the equations of motion of Ref. 1 also are incorrect.

The corrected equations, after dropping the second term of Eq. (1) for force and torque, are as follows:

$$\bar{l} = m(\dot{\bar{R}} + \dot{\bar{\rho}}) \tag{9}$$

$$\bar{F} = m(\ddot{\bar{R}} + \ddot{\bar{\rho}}) + \dot{m}(\dot{\bar{R}} + \dot{\bar{\rho}}) \tag{10}$$

$$\bar{T}_b = \bar{\rho} \times [(\ddot{\bar{R}} + \ddot{\bar{\rho}})m + \dot{m}(\dot{\bar{R}} + \dot{\bar{\rho}})] \tag{11}$$

Discussion

Note that $\dot{\bar{R}}$ gives the velocity of the origin of body-axes frame and $\dot{\bar{\rho}}$ gives the velocity of exiting or entering fluid mass (with mass flow rate $\dot{m}$) with reference to the origin of the body frame. The quantity $\dot{m}(\dot{\bar{R}} + \dot{\bar{\rho}})$ gives the negative net thrust acting on the moving body. If there is no inlet, then that quantity is equal to the negative thrust from the exhaust gases of the rocket. Thus, if thrust is considered as an external force, one must not again include the terms attributable to $\dot{m}$. On the other hand, if $\dot{m}$ terms are included, one must not consider thrust as an external force acting on the vehicle.

In the torque equation, the term $\bar{\rho} \times (\dot{\bar{R}} + \dot{\bar{\rho}})\dot{m}$ gives the negative torque about the origin of the body frame attributable to thrust. Here again, one should retain either the torque attributable to the $\dot{m}$ term or the external torque attributable to thrust, but not both.

Reference 1 has retained both terms, which is incorrect.

The thrust values specified for the air-breathing engine could include the effect of $\dot{m}_{\text{air}}$ at inlet or the values could be taken after removal of the effect of $\dot{m}_{\text{air}}$. If it is the latter, the corresponding terms at the inlet of air must be retained.

The following additional comments can be made on the paper.

Force should not depend on the location of the origin of the coordinate system. Terms containing $\ddot{m}_{\text{fuel}}$ from the equations in Table 1 disappear if the origin is taken at the outlet, which indicates an anomaly.

Section V on the point-mass model contains the statement "the fluid flow terms which depend primarily on the vehicle angular velocity $W_{B,I}$ do not appear in Table 4 because the point mass model does not include rotational dynamics." This statement is incorrect because the forward acceleration equation (Table 1) contains terms $2\dot{m}_{\text{air}}(\dot{x}_{\text{out}} - \dot{x}_{\text{in}})$ and $2\dot{m}_{\text{fuel}}\dot{x}_{\text{out}}$. The statement, in addition to the negligible effect of $\dot{m}$ terms on the final results reported by the authors, gives us a feeling that the authors are treating $\dot{x}_{\text{in}}$ and $\dot{x}_{\text{out}}$ as numerically zero velocities for the location of inlet and outlet with reference to the origin instead of treating them as the velocity of

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air/fluid mass elements. This is not a correct interpretation because (x, y, z) are the coordinates of the mass element and their time derivatives should be the velocity of the fluid mass at inlet and the outlet.

To conclude, there is no need to treat the air-breathing engine separately by including $\dot{m}$ terms. Its effect is completely accounted for by using the thrust attributable to the engine [which is equal to $-\dot{m}(\dot{\bar{R}}+\dot{\bar{\rho}})$] for computing components of external force and torque to be used on the right-hand side of equations of motion given in Tables 1 and 2 of Ref. 1.

Reference

[1]Bilimoria, K. D., and Schmidt, D. K., "Integrated Development of the Equations of Motion for Elastic Hypersonic Flight Vehicles," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 1, 1995, pp. 73–81.

Reply by the Authors to Nivritti Vithoba Kadam

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Introduction

IN our paper,[1] a Lagrangian approach is used to determine the dynamics of rigid-body motion, elastic deformation, fluid flow, rotating machinery, wind, and a spherical rotating Earth model and to account for their mutual interactions. The preceding Technical Comment claims that the terms containing mass flow rates in the equations of motion of Ref. 1 are erroneous and presents an analysis using a Newtonian approach to support this claim. We stand by the equations of motion presented in Ref. 1 and note that the claim and many other statements made in the Technical Comment arise primarily from a misinterpretation of certain velocity terms.

Contents

Using a Newtonian approach, Meriam[2] gives a derivation of force and moment equations for variable-mass systems. For the case where the mass of the system is distributed over a finite volume and the system may have any general motion, the force equation is given by Eq. (173) of Ref. 2, reproduced as follows:

$$\sum \bar{F} = m\ddot{\bar{r}}_e - \dot{m}\bar{u} - \frac{d^2}{dt^2}(m\bar{\rho}_e) \tag{1}$$

Equation (1) is written in the nomenclature of Ref. 2; using the nomenclature of Ref. 1, this equation is written as

$$m\bar{g} + \bar{F}_A + A_{\text{open}}(P - P_\infty)\hat{n}_{\text{open}} = m\ddot{\bar{E}}_{\text{open}} - \dot{m}\bar{V}_{f,0} - \frac{d^2}{dt^2}(m\bar{r}_{\text{open}}) \tag{2}$$

Noting that $\dot{m} = \dot{m}_{\text{fluid}}$, $\bar{E}_{\text{open}} = \bar{R} + \bar{r}_{\text{open}}$, and expanding the last term on the right-hand side of Eq. (2) results in

$$m\bar{g} + \bar{F}_A + A_{\text{open}}(P - P_\infty)\hat{n}_{\text{open}} = m\ddot{\bar{R}} - \dot{m}_{\text{fluid}}\bar{V}_{f,0} - 2\dot{m}_{\text{fluid}}\dot{\bar{r}}_{\text{open}} - \ddot{m}_{\text{fluid}}\bar{r}_{\text{open}} \tag{3}$$

Noting that the time derivatives of $\bar{R}$ and $\bar{r}_{\text{open}}$ are evaluated in the inertial frame, that $\bar{V}_I = (d\bar{R}/dt)|_I$, and that fluid flow takes place through several openings in the vehicle, one obtains

$$m\bar{g} + \bar{F}_A + \sum_{\text{openings}} [A_{\text{open}}(P - P_\infty)\hat{n}_{\text{open}} + \dot{m}_{\text{fluid}}\bar{V}_{f,0}] = m\frac{d\bar{V}_I}{dt}\bigg|_I - \sum_{\text{openings}}\left[2\dot{m}_{\text{fluid}}\frac{d\bar{r}_{\text{open}}}{dt}\bigg|_I + \ddot{m}_{\text{fluid}}\bar{r}_{\text{open}}\right] \tag{4}$$

Using the definition of $\bar{F}_T$ given by Eq. (35) of Ref. 1 and applying Eq. (1) of Ref. 1 to $\bar{r}_{\text{open}}$, Eq. (4) can be written as

$$m\frac{d\bar{V}_I}{dt}\bigg|_I = m\bar{g} + \bar{F}_A + \bar{F}_T + \sum_{\text{openings}}\left[2\dot{m}_{\text{fluid}}\left(\frac{d\bar{r}_{\text{open}}}{dt}\bigg|_B + \bar{\omega}_{B,I}\times\bar{r}_{\text{open}}\right) + \ddot{m}_{\text{fluid}}\bar{r}_{\text{open}}\right] \tag{5}$$

Equation (5) is the same as Eq. (41) of Ref. 1, which was independently developed using a Lagrangian approach. The left-hand side of this equation can be expanded as described in Ref. 1; the resulting vector force equation can be written as the three scalar force equations presented in Table 1 of Ref. 1.

Similarly, starting from the vector moment equation given by Eq. (174) of Ref. 2, we obtain Eq. (46) of Ref. 1, which in turn leads to the three scalar moment equations presented in Table 2 of Ref. 1.

Reference 1 defines the quantity $\bar{r}_{\text{open}}$ as the average location of an opening (inlet or outlet) in the vehicle, relative to the origin of the body frame (vehicle center of mass). The time derivative of $\bar{r}_{\text{open}}$ evaluated in the body frame, $(d\bar{r}_{\text{open}}/dt)|_B$, is the velocity of the opening relative to the body frame; this term arises from elastic motion of the vehicle, and from changes in vehicle c.m. location because of fuel consumption. The body axes scalar components of the vector $(d\bar{r}_{\text{open}}/dt)|_B$ for inlets and outlets are $(\dot{x}_{\text{in}}, \dot{y}_{\text{in}}, \dot{z}_{\text{in}})$ and $(\dot{x}_{\text{out}}, \dot{y}_{\text{out}}, \dot{z}_{\text{out}})$, respectively. It is emphasized that these quantities represent velocities because of the elastic motion of the vehicle mass elements at openings and changes in vehicle c.m. location because of fuel consumption; they are not velocities of fluid mass elements at openings.

As defined in Ref. 1, the velocity of fluid mass elements relative to an opening is $\bar{V}_{f,0}$. It is noted that this velocity only appears implicitly in the force and moment equations presented in Ref. 1 through the thrust terms $\bar{F}_T$ and $\bar{M}_T$ defined in Eqs. (35) and (38), respectively, of Ref. 1.

There is no anomaly indicated by the observation that terms containing $\dot{m}_{\text{fluid}}$ will disappear if the origin is taken at an opening. Reference 1 defines the origin of the body frame at the instantaneous mass center of the vehicle. Also, Ref. 2 states that "the linear momentum of a time-varying mass depends on the location with respect to the mass center of the position where mass is added (or subtracted)."

In the development of the point-mass model, the explicitly appearing fluid flow terms in the force equation [see Eq. (5)] have been dropped. The primary contribution of these terms results from the vehicle angular velocity $\bar{\omega}_{B,I}$, which is not defined in the point-mass model; the secondary contributions resulting from elastic effects, changes in vehicle c.m. location because of fuel consumption, and unsteady mass flow effects have been neglected. The mass flow rate term implicit in the thrust force $\bar{F}_T$ has, of course, been retained.

Finally, the corrected force and moment equations derived in the Technical Comment [Eqs. (10) and (11)] are themselves erroneous. For a variable mass system, the resultant of external forces/moments is not just equal to the time rate of change of linear/angular momentum as stated [see Eq. (2) and an unnumbered equation preceding Eq. (3)] in the Technical Comment. The resultant of external forces/moments equals the time rate of change of the linear/angular momentum of the varying mass minus the rate at which linear/angular momentum is being changed by the mass elements entering and leaving the system; see Eqs. (171) and (174) of Ref. 2. This principle is also used in the derivation of the Navier–Stokes equations,[3] which are based on conservation of momentum and are consistent with Newton's laws.

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